

DISCRETE MATHEMATICS: COMBINATORICS AND GRAPH THEORY

Practice Exam 3

Instructions. Solve any 5 questions and state which 5 you would like graded. Note that this is a sample exam, and while it bears some similarity to the real exam, the two are not isomorphic.

1. Solve the following recurrence relations:

(a) $a_n = 6a_{n-1} - 9a_{n-2}$ when $a_0 = 2, a_1 = 21$.

(b) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$ with $a_0 = 3, a_1 = 6$, and $a_2 = 0$.

(c) $a_n = 2a_{n-1} + 1$ when $a_1 = 1$.

(d) $na_n = (n - 2)a_{n-1} + 2$ when $a_1 = 1$.

(e) $a_n = 3a_{n-1} + 10a_{n-2} + 7.5n$ where $a_0 = 4$ and $a_1 = 3$.

2. Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has

(a) exactly three boys?

(b) at least one boy?

(c) at least one girl?

(d) all children of the same sex?

3. How many members of the set $S = \{1, 2, 3, \dots, 105\}$ have nontrivial factors in common with 105? Hint: use the inclusion-exclusion principle.

4. Suppose that 8% of the patients tested in a clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that

(a) a patient testing positive for HIV with this test is infected with it?

(b) a patient testing positive for HIV with this test is not infected with it?

(c) a patient testing negative for HIV with this test is infected with it?

(d) a patient testing negative for HIV with this test is not infected with it?

5. Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one, two or three stairs at a time. What are the initial conditions? How many ways can this person climb a flight of eight stairs?

6. Show that $\mathbb{E}_Y(\mathbb{E}_X(P(X|Y))) = X$

7. Consider a random walk (a drunk stumbling in one dimension) with step sizes of S_i where S_i is $+1$ with probability p and -2 with probability $q = 1 - p$. Let $T_n = \sum_{i=1}^n S_i$ be the displacement after a fixed (not random) number of steps n . Find the probability $P(T_n = t)$ and the mean and variance of T_n in terms of n and p .